Observer Design for a Class of Nonlinear Systems with Bad Trajectories

Jaime A. Moreno*

Instituto de Ingeniería, Universidad Nacional Autónoma de México (UNAM),
Ap. Postal 70-472, 04510, México, D.F., México.
JMorenoP@iingen.unam.mx

Abstract

A necessary condition for the existence of global observers is the detectability of the plant, i.e. the convergence of indistinguishable trajectories. If non converging indistinguishable trajectories exist, i.e. so called bad trajectories, then no (strong) observer exists. In this paper weak observers are proposed for a class of systems that can have bad trajectories. The convergence is assured for a set of plant's trajectories that excludes the bad ones. This set is characterized by conditions on the plant's trajectories that resembles and generalizes the persistently exciting conditions, well-known in the identification and adaptive control literature.

1 Introduction

An intensive research activity has been done in the last years aiming at developing design strategies for nonlinear observers, and different methods have been proposed [10, 12].

One special difficulty with nonlinear systems is that the observability/detectability properties are input (trajectory) dependent, i.e. two initial states can be distinguishable using some input function but indistinguishable with other input functions, so called *bad inputs* [13]. This constitutes a kind of singular situation for the observation problem [4], and the existence of bad inputs makes it very difficult to design an observer. Most observer design methods exclude systems with bad inputs: The existence of high-gain observers requires uniform observability for every input [6], and the classical error linearization methods [9] impose even stronger conditions on the system's observability properties. The detectability of nonlinear systems has been studied in different aspects, especially in the context of the relationship between input/output and Lyapunov stability. However, the relationship between detectability and the observer design problem has not been yet completely clarified and only some partial results have been obtained [**?**, 14]. Systems with bad trajectories constitute an important class. For example, the parameter identification problem leads to observation problems for this kind of systems. Most known observer design results for such systems are derived for those affine in the unmeasured variables [4, 7, 12], or transformable to them. In this case the observer can be basically designed as one for a linear time varying system. The required uniform observability conditions for the linear system correspond then to the persistence of excitation conditions, originally introduced for parameter identification. However, for systems nonlinear in the unmeasured variables almost no results are known. Sometimes they can be transformed to the affine case by immersion in a higher dimensional system, but this leads to problems similar to those caused by overparametrization in identification and adaptive control. So it is important to treat the general case in a direct manner, as it has been done recently for the identification of nonlinearly parameterized systems [3].

The objective of this paper is to propose a method to design observers for a class of systems that can have bad trajectories, and the unmeasured variables are not required to enter linearly. The system class and the design idea, based on the use of the circle criterion, constitute a generalization of those proposed by [1, 5]. However, despite of the similarities, the objectives are completely different: [1, 5] aims at the elimination of the linear growth assumption for the nonlinearities and the system class is so restricted, that bad trajectories are excluded. Our objective is the design of observers for systems with bad trajectories, and the class of systems is extended to allow for these singularities. For simplicity only

*On leave at the ISR, University of Stuttgart, Germany. This work has been done with the financial aid of DGAPA-UNAM under project PAPIIT IN106002-2 and CONACyT under project 34934A.

ISBN: 970-32-2137-8

the global case will be considered.

The rest of the paper is organized in the following form. In the next Section a preliminary discussion is given on the class of observers that are meaningful for systems with bad trajectories. The class of systems to be considered in this work is introduced in Section 3. This class includes systems with bad trajectories and that are nonlinear in the unmeasured variables. The proposed method is described in Section 4. In Section 5 an illustrative example is presented.

2 Detectability and Existence of Observers

Consider the nonlinear system

$$\tilde{\Sigma}:\left\{\begin{array}{l}\dot{x}=f\left(x,u\right)\ ,\quad x\left(0\right)=x_0\\ y=h\left(x\right)\ ,\end{array}\right. \tag{1}$$

where $x\in\mathbb{R}^n$ is the state vector, $u\in\mathbb{R}^p$ is the input vector, and $y\in\mathbb{R}^m$ is the output vector. f, and h are sufficiently smooth so that existence and uniqueness of solutions is guaranteed. The solution of Σ which passes through x_0 at $t=0$, corresponding to the input function $u\left(\cdot\right)$, is denoted as $x\left(t,x_0,u\left(\cdot\right)\right)$, and $y\left(t,x_0,u\left(\cdot\right)\right)=h\left(x\left(t,x_0,u\left(\cdot\right)\right)\right)$ represents the output. If there is no confusion they will be written as $x\left(t\right)$ and $y\left(t\right)$.

If for $\tilde{\Sigma}$ an input $u\left(\cdot\right)$ and two initial conditions $\bar{x}$, x are given, such that $y\left(t,x,u\left(\cdot\right)\right)=y\left(t,\bar{x},u\left(\cdot\right)\right)$, for all $t\geq 0$, then $\bar{x}$ and x are $u\left(\cdot\right)$-*indistinguishable*. Denote by $\mathcal{I}_{(u,x)}$ the set of all $u\left(\cdot\right)$-indistinguishable states from x. $\tilde{\Sigma}$ is *observable* if for every x, and every $u\left(\cdot\right)$ it is satisfied that $\mathcal{I}_{(u,x)}=\{x\}$. $\tilde{\Sigma}$ is *detectable* if for every x, if $\bar{x}\in\mathcal{I}_{(u,x)}$ then their trajectories converge, i.e. $\lim_{t\to\infty}\|x(t,\bar{x},u\left(t\right))-x(t,x,u\left(t\right))\|=0$. If $\tilde{\Sigma}$ is not detectable, then there exist diverging indistinguishable, i.e. *bad trajectories*, which correspond to some *bad input*.

An *observer* for $\tilde{\Sigma}$ is a dynamical system Ω that has as inputs the input u and the output y of $\tilde{\Sigma}$, and its output $\hat{x}$ is an estimation of the state x of $\tilde{\Sigma}$. Ω is a *strong observer* if there is some initial condition of Ω such that for *every* trajectory of $\tilde{\Sigma}$ the estimation $\hat{x}$ converges to the state x, i.e. $\lim_{t\to\infty}\|\hat{x}(t)-x(t)\|=0$. Ω is a *weak observer* if the convergence is only assured for a proper subset of the trajectories of $\tilde{\Sigma}$.

Note that local observers are a kind of weak ones. However, our interest here is more on global observers that exclude some plant trajectories. The following result is a simple consequence of the definitions.

Lemma 1 *If $\tilde{\Sigma}$ has a* strong *observer, then $\tilde{\Sigma}$ is detectable. I.e. if $\tilde{\Sigma}$ has bad inputs (trajectories) then it does not have a strong observer.*

The importance of this simple result is that in the design of observers for system with bad inputs it is indispensable to allow the observer not to converge for some system trajectories, including the set of bad ones. Since the aim of most observer design methods is the design of strong observers, they have to exclude the class of systems with bad trajectories.

Since for linear systems the lack of detectability excludes the possibility of designing any reasonable observer (the set of excluded trajectories is very big), a comment is in order to explain why this is not necessarily the case for nonlinear systems. Since for analytical systems the set of bad inputs is generically very small [13], then one can reasonably expect that the set of trajectories that has to be excluded from the set of those leading to the convergence of the observer is also small. This is confirmed in the literature of observer design for systems linear in the unmeasured states [4, 7] and in the adaptive control and identification literature [11]. In these cases the set of trajectories for which the observer converge is characterized by means of persistence of excitation conditions. Weak observers constitute therefore an important class of observers, and in the following Section a method is proposed to design them for a class of systems. The class of trajectories for which the observer is assured to converge will be characterized by generalized persistence of excitation.

3 Problem Formulation

Consider a plant that can be brought to the form

$$\Sigma:\left\{\begin{array}{l}\dot{x}=Ax+G\psi\left(\sigma,u\right)+\varphi\left(t,y,u\right)\ ,\quad x\left(0\right)=x_0\\ y=Cx\ ,\\ \sigma=Hx\end{array}\right. \tag{2}$$

where x, u, y are as for (1), and $\sigma\in\mathbb{R}^r$ is a (not necessarily measured) linear function of the state. $\varphi\left(t,y,u\right)$ is an arbitrary nonlinear function of the time, the input and the output. $\psi\left(\sigma,u\right)$ is an r-dimensional vector that depends on the input and σ. ψ and φ are assumed to be locally Lipschitz in σ or y, continuous in u, and piecewise continuous in t. Since the plant Σ is not assumed globally Lipschitz the global existence of solutions is not guaranteed, i.e. for some initial conditions and inputs finite escape time is possible. This is a not desirable situation

ISBN: 970-32-2137-8

and will be excluded by assuming that Σ (2) is either *complete*, i.e. the state trajectories $x(t)$ are defined for every $t \geq 0$, every initial condition $x_0 \in \mathbb{R}^n$ and every input $u(\cdot) \in \mathcal{U}$, or the initial states and/or inputs are so restricted that the state trajectory is locally bounded, i.e. $x(t) \in \mathcal{L}_{\infty e}$. This class includes the one proposed in [1], but it includes systems with bad trajectories (see the example in Section 5).

An (strong or weak) observer for Σ of the form

$$\Omega : \begin{cases} \dot{\hat{x}} = A\hat{x} + L(y-\hat{y}) + G\psi(\hat{\sigma} + N(y-\hat{y}), u) + \\ \qquad + \varphi(t,y,u) \ , \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y} = C\hat{x} \ , \\ \hat{\sigma} = H\hat{x} \end{cases} \tag{3}$$

is proposed, where matrices $L \in \mathbb{R}^{n\times p}$, and $N \in \mathbb{R}^{r\times p}$ have to be designed. Defining the state estimation error $e \triangleq x - \hat{x}$, the output estimation error $\tilde{y} \triangleq y - \hat{y}$, and the function estimation error $\tilde{\sigma} \triangleq \sigma - \hat{\sigma}$, the dynamics of e is given by

$$\begin{aligned} &\dot{e} = (A - LC)e + G[\psi(\sigma,u) - \psi(\hat{\sigma} + N\tilde{y}, u)] \ , \\ &\tilde{y} = Ce \ , \qquad e(0) = e_0 = x_0 - \hat{x}_0 \\ &\tilde{\sigma} = He \ . \end{aligned}$$

Note that $\hat{\sigma} + N\tilde{y} = H\hat{x} + NCe = Hx - He + NCe = \sigma - (H - NC)e$. Defining

$$\begin{aligned} z &\triangleq (H - NC)e = \tilde{\sigma} - N\tilde{y} \\ \phi(z,\sigma,u) &\triangleq \psi(\sigma,u) - \psi(\sigma - z, u) \ , \end{aligned} \tag{4}$$

the dynamics of the error can be written as

$$\Xi : \begin{cases} \dot{e} = A_L e + G\phi(z,\sigma,u) \ , \quad e(0) = e_0 \\ z = H_N e \ , \end{cases} \tag{5}$$

where $A_L \triangleq A - LC$, and $H_N \triangleq H - NC$. Note that $\phi(0,\sigma,u) = 0$ for all σ and u. The error dynamics (5) is not autonomous, as in the linear case, but it is driven by the plant (2) through its input u, and the linear function of the state $\sigma = Hx$. ϕ is therefore a time varying nonlinearity, whose time variation depends on the input/state trajectory of the plant. It will be assumed that the memoryless function $\phi(z,\sigma,u)$, that is given by the problem data, belongs to a sector $[K_1, K_2]$ with respect to z, where $K_1, K_2 \in \mathbb{R}^{r\times r}$ are constant, quadratic matrices, such that $K \triangleq K_2 - K_1 = K^T > 0$, i.e. K is a positive definite symmetric matrix. This means that (see [8])

$$[\phi(z,\sigma,u) - K_1 z]^T [\phi(z,\sigma,u) - K_2 z] \leq 0 \tag{6}$$

for all (z,σ,u). In case $K_2 = \infty$, i.e. for a non Lipschitz nonlinearity, the sector condition (6) becomes $z^T[\phi(z,\sigma,u) - K_1 z] \geq 0$. If the inequality is strict then the sector is written as $(K_1, K_2]$, $[K_1, K_2)$, or (K_1, K_2). Note that a sector is a conic subset of $\mathbb{R}^{2r}$.

4 Observer Design Method

The error dynamics Ξ (5) is a feedback connection of a LTI system and a time-varying non linearity. For these kinds of systems the circle criterion [8] is a *sufficient* asymptotic stability condition for nonlinearities in a sector. The sector of stability is a property of the LTI subsystem, i.e. given the system matrices (H_N, A_L, G) there is a *stability sector* (K_1^a, K_2^a) such that Ξ has $e = 0$ as a global asymptotically stable equilibrium point for every nonlinearity $\phi(z,t)$ belonging to any sector contained in the stability sector, i.e. $\phi \in [K_1, K_2] \subset (K_1^a, K_2^a)$. Note that (K_1^a, K_2^a) is a function of the matrices L and N.

4.1 Strong observers

If the Σ is detectable, then one can expect to be able to design a *strong* observer. If there exist L and N such that the stability sector of Ξ (5) contains the nonlinearity sector $[K_1, K_2]$, then Ω given by (3) is an observer for the plant Σ, that converges exponentially fast for *every* (locally bounded) system trajectory. Theorem 2 is an improvement of Theorem 1 in [1].

Theorem 2 *Consider the plant Σ (2). If there exist matrices L and N such that*

a) If $\phi \in [K_1, \infty]$

$$\begin{matrix} P\tilde{A}_L + \tilde{A}_L^T P = -R^T R - \epsilon P \\ PG = -H_N^T \end{matrix} \ , \tag{7}$$

b) If $\phi \in [K_1, K_2]$, with $K = K_2 - K_1 = K^T > 0$,

$$\begin{matrix} P\tilde{A}_L + \tilde{A}_L^T P = -R^T R - \epsilon P \\ PG = -H_N^T K^T - R^T W \\ W^T W = I \end{matrix} \ ,$$

with $\tilde{A}_L = A_L + GK_1H_N$, are satisfied for some $P = P^T > 0$, $\epsilon > 0$, and matrices R, W, then, there exist constants $\kappa, \beta > 0$ such that for Ξ (5)

$$\|e(t)\| \leq \kappa \|e(0)\| \exp(-\beta t) \ , \ \forall t \geq 0 \ ,$$

for every locally bounded trajectory of the plant, i.e. system Ω (3) is a strong *observer for the plant.*

Proof. Only the case (a) will be considered. Case (b) follows the same path. Consider that (7) is satisfied. Take $V(e) = (1/2)e^T P e$ as a Lyapunov function candidate for the observation error (5). Its time derivative along their trajectories is

$$\begin{aligned} \dot{V}(e) &= \tfrac{1}{2}e^T\left[PA_L + A_L^T P\right]e + e^T PG\phi(z,\sigma,u) \\ &= \tfrac{1}{2}e^T\left[P\tilde{A}_L + \tilde{A}_L^T P\right]e + e^T PG(\phi - K_1 H_N\, e) \\ &= -\epsilon V(e) - \tfrac{1}{2}e^T R^T R e - e^T H_N^T(\phi - K_1 H_N\, e) \\ &\leq -\epsilon V(e) - z^T(\phi - K_1 z) \leq -\epsilon V(e) \ , \end{aligned}$$

ISBN: 970-32-2137-8

since $\phi \in [K_1, \infty]$. The conclusion follows. ■

Note also, that no observability or controllability is assumed for the linear system in Ξ, in contrast to the circle criterion.

4.2 Weak observers

If system Σ does have bad trajectories, then because of Lemma 1 no *strong* observer exists, and Theorem 2 cannot be satisfied. However, the following two step procedure can lead to a *weak* observer. For simplicity, only the special case when the nonlinearity belongs to the sector $\phi \in [K_1, \infty]$ will be considered, but the same principle applies in general.

4.2.1 Sector inclusion

Selecting matrices L and N the stability sector of the LTI system (H_N, A_L, G) can be adjusted. Once again for simplicity, only stability sectors $(K_1^a, \infty]$ with infinity upper bound will be considered. Since the strong observer condition $[K_1, \infty] \subset (K_1^a, \infty]$, that is equivalent to the matrix inequality $K_1^a(L,N) < K_1$, cannot be satisfied, a reasonable objective is to maximize the intersection between the sector of the nonlinearity $[K_1, \infty]$ and the stability sector of the LTI system $(K_1^a, \infty]$. So L and N will be selected such that

$$\min_{L,N} \lambda_{\min} \left\{ K_1^a(L,N) + K_1^{aT} - K_1 - K_1^T - \delta I \right\} \ , \tag{8}$$

where $\lambda_{\min}\{M\}$ is the minimum value of matrix M, and $\delta > 0$ is a small regularization term for the minimization problem. This term is necessary since K_1^a satisfies (7) with $\epsilon = 0$, and no asymptotic stability is assured. The value of K_1^a obtained by the minimization will be called $\tilde{K}_1$.

4.2.2 Conditioned convergence

The circle criterion assures stability of the observer Ω (3) only for nonlinearities in the sector $\left[\tilde{K}_1, \infty\right] \subset [K_1, \infty]$. Since the nonlinearity $\phi(z, \sigma, u)$ depends on the plant signals (σ, u), external to the error system Ξ, by restricting the values of (σ, u) it is in general possible to maintain ϕ in the sector $\left[\tilde{K}_1, \infty\right]$, guaranteeing observer convergence. This condition imposes a strong restriction on the plant trajectories. However, signals (σ, u) can take values for which $\phi(\cdot, \sigma, u) \notin \left[\tilde{K}_1, \infty\right]$ as long as this sector violation is only temporal. System Ω (3) becomes then a *weak* observer, whose convergence is dependent on the values of the plant's variables (σ, u).

The convergence conditions can be found by a Lyapunov analysis. Note that system (H_N, A_L, G) satisfies (7) for the sector $\left[\tilde{K}_1, \infty\right]$ for some matrices L and N. Taking the derivative of $V(e) = (1/2)\, e^T P e$ along the trajectories of Ξ is

$$\dot{V}(e) \leq -\epsilon V(e) - z^T \left(\phi - \tilde{K}_1 z\right) \ . \tag{9}$$

Since $\phi \in [K_1, \infty]$, the term $z^T \left(\phi - \tilde{K}_1 z\right)$ is bounded below by a quadratic form in z, i.e.

$$z^T \left(\phi - \tilde{K}_1 z\right) \geq \theta(\sigma, u) \|z\|^2 \ , \tag{10}$$

and $\theta(\sigma, u) \geq b$, a constant. For values of (σ, u) such that $\theta(\sigma, u) \geq 0$ the nonlinearity $\phi \in \left[\tilde{K}_1, \infty\right]$, whereas when $\theta(\sigma, u) < 0$ the nonlinearity $\phi(\cdot, \sigma, u)$ leaves the sector. If for all (σ, u), $\theta(\sigma, u) \geq 0$ then the strong observer case in Theorem 2 is recovered.

Using (10) in (9)

$$\dot{V} \leq -\epsilon V - \theta(\sigma, u) \|z\|^2 \leq -\left[\epsilon - \varrho(\sigma, u)\right] V \ ,$$

where

$$\varrho(\sigma, u) \triangleq -\frac{\lambda_H}{\lambda_P} \theta(\sigma, u) \left\{ \begin{array}{lll} 1 & \text{if} & \theta(\sigma, u) < 0 \\ 0 & \text{if} & \theta(\sigma, u) \geq 0 \end{array} \right. \ , \tag{11}$$

and λ_m, λ_M and λ_H are the minimal and maximal eigenvalues of P and the maximal eigenvalue of $H_N^T H_N$, respectively. By the comparison lemma [8] it follows that

$$V(t) \leq \exp\left[-\eta(t, t_0)\right] V(t_0) \ ,$$

where

$$\eta(t, t_0) \triangleq \epsilon(t - t_0) - \int_{t_0}^{t} \varrho(\sigma(\tau), u(\tau))\, d\tau \ . \tag{12}$$

Note that if

$$\lim_{t \to \infty} \eta(t, t_0) = \infty \ , \tag{13}$$

then $\lim_{t \to \infty} \|e(t)\| = 0$, and so the estimation error converges to zero. However, this convergence is not necessarily uniform. If instead it is satisfied

$$\int_{t_0}^{t} \varrho(\sigma(\tau), u(\tau))\, d\tau \leq \alpha(t - t_0) + \gamma \tag{14}$$

with $\alpha, \gamma \in \mathbb{R}$, and $\alpha < \epsilon$, then

$$\begin{aligned} V(t) &\leq \exp\left[-\eta(t, t_0)\right] V(t_0) \\ &\leq \exp[\gamma] \exp\left[-(\epsilon - \alpha)(t - t_0)\right] V(t_0) \end{aligned}$$

and therefore

$$\|e(t)\| \leq \sqrt{\frac{\lambda_M \exp[\gamma]}{\lambda_m}} \exp\left[-\frac{(\epsilon - \alpha)}{2}(t - t_0)\right] \|e_0\| \ ,$$

i.e. the estimation error converges exponentially to zero. This proves the following theorem

ISBN: 970-32-2137-8

Theorem 3 *Consider Σ (2). Suppose that there exist matrices L and N such that equations (7) are satisfied for some $P = P^T > 0$, $\epsilon > 0$, R, and $\tilde{K}_1$, that solves (8). Suppose that ϕ belongs to the sector $\phi \in [K_1, \infty]$. If condition (13) is satisfied, where $\eta(t, t_0)$ is defined in (12), then the estimation error converges to zero, i.e. $\lim_{t\to\infty} \|e(t)\| = 0$. If, moreover, condition (14) is satisfied, such that $\alpha, \gamma \in \mathbb{R}$, and $\alpha < \epsilon$, then there exist positive constants κ, and β such that*

$$\|e(t)\| \leq \kappa \|e(t_0)\| \exp(-\beta(t - t_0)) \ , \ \forall t \geq t_0 \ .$$

In both cases Ω (3) is a weak *observer for Σ.*

Remark 4 *Theorem 3 is a generalization of theorem 2. The latter corresponds to the former when α, and γ can be set to zero.*

Remark 5 *Condition (14) is stronger than condition (13). Since the values of $\eta(t, t_0)$ depend on the trajectories of the plant Σ, these conditions constitute a restriction on the set of system's trajectories, i.e. trajectories that satisfy such conditions are contained in a subset (a proper one if there are diverging bad inputs) of the set of plant's trajectories. These conditions are similar to the persistence of excitation conditions well known in the parameter identification and adaptive control literature [11], or similar conditions in the observer literature [2, 7].*

Remark 6 *Recall that if the nonlinearity is in the stability sector then condition (14) is satisfied. However, this condition allows the nonlinearity to abandon this sector, and global exponential stability is still assured as long as the condition is satisfied. If only condition (13) is satisfied, then not stability but just convergence is assured. Some conditions that imply condition (14) are (see [8, Lemma 9.5]):*

1. *If*
$$\int_0^\infty \varrho(\sigma(\tau), u(\tau))\, d\tau \leq k$$
then (14) is satisfied with $\alpha = 0$, and $\gamma = k$.

2. *If*
$$\varrho(\sigma(t), u(t)) \to 0 \quad as \quad t \to \infty$$
then for any $\alpha > 0$, there is $\gamma = \gamma(\alpha) > 0$ such that (14) is satisfied.

3. *If there are constants $\Delta > 0$, $T \geq 0$, and $\alpha_1 > 0$ such that*
$$\frac{1}{\Delta}\int_t^{t+\Delta} \varrho(\sigma(\tau), u(\tau))\, d\tau \leq \alpha_1 \ , \quad \forall t \geq T$$
then (14) is satisfied with $\alpha = \alpha_1$, and $\gamma = \alpha_1\Delta + \int_0^T \varrho(\sigma(\tau), u(\tau))\, d\tau$.

5 Example

For illustration consider the system

$$\Sigma : \begin{cases} \dot{x}_1 = -x_1 + g(x_2) \ , \\ \dot{x}_2 = -ax_2 + u \ , \\ y = x_1 \ , \end{cases}$$

with $g(x_2) = x_2\left(x_2^2 - 1\right)$. This system has bad inputs for $a = 0$, since $u = 0$ makes indistinguishable, for example, the initial states $(x_{10}, 0)$ and $(x_{10}, 1)$, and their trajectories do not converge to each other, i.e. Σ is not detectable. However, for $a > 0$ the system is detectable and does not have bad inputs. This system can be written in the form (2), and the error equation (5) is given with

$$A_L = \begin{bmatrix} -1 - l_1 & 1 \\ -l_2 & -a \end{bmatrix}, G = C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H_N^T = \begin{bmatrix} -k \\ 1 \end{bmatrix}$$
$$\sigma = x_2 \ , \ \phi(z, \sigma) = z\left(z^2 - 3\sigma z + 3\sigma^2 - 2\right) \ .$$

For fixed σ the nonlinearity belongs to the sector $\phi \in \left[3\sigma^2/4 - 2, \infty\right]$, and to the sector $\phi \in [-2, \infty]$ for all the values of σ. Note that the sector of the nonlinearity depends on the values taken by the plant state $\sigma = x_2$. If, for example, the initial states and the input to the plant are so selected that for all $t \geq 0$, $|x_2(t)| \geq 2$, then the resulting nonlinearity $\phi(z, \sigma(t))$ will stay in the sector $[1, \infty]$.

For this simple SISO case the satisfaction of the circle criterium can be interpreted in the frequency domain. The transfer function of the linear part of the error equation is given by

$$G(s) = \frac{k(s + a) + l_2}{s^2 + (l_1 + a + 1)s + (l_1 + 1)a + l_2} \ .$$

In this case the satisfaction of the circle criterion for the sector $[\varrho, \infty]$ corresponds to the selection of the parameters (k, l_1, l_2) so that the loop transformed transfer matrix $H(s) = G(s)/(1 + \varrho G(s))$ is SPR [8]. This is the case iff both numerator and denominator polynomials of $H(s)$ are Hurwitz and $(\varrho k + l_1 + a + 1) > a + l_2/k > 0$. If $a > 0$ it is possible to satisfy the circle criterion for $\varrho = -2$, i.e. the designed observer converge for every trajectory of the system, i.e. it is a strong one.

If $a = 0$, the plant is not detectable, but a weak observer can be designed. If in (8) $\delta = 0$ is set, the maximal stability sector of the LTI system is found to be $(K_1^a, \infty] = (-1, \infty]$, and so the nonlinearity sector cannot be completely covered. If the trajectories of the plant are so restricted, that $\sigma = x_2$ satisfies $|\sigma| > 2/\sqrt{3}$, then the nonlinearity will stay in the sector $(-1, \infty]$ and the observer will converge. However, if σ takes values $|\sigma| \leq 2/\sqrt{3}$, then the observer will converge if the "persistence of excitation"

ISBN: 970-32-2137-8

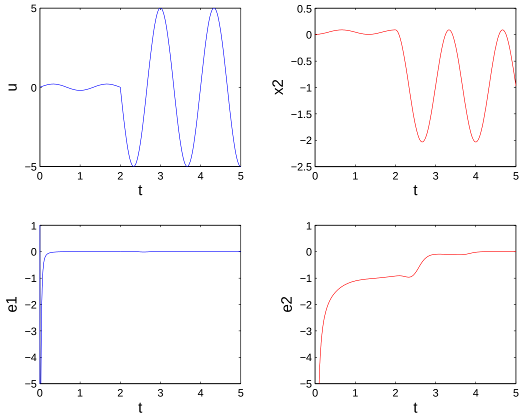

Figure 1: Simulation results for the example with the following parameters: $a = 0$, $k = 2$, $l_1 = 200$, $l_2 = 120$, $x_0 = 0$, $e_0 = (70, -50)$. Shown are the input applied u, the state variable x_2, and the estimation errors e_1, e_2.

condition (14) is satisfied, i.e. if σ does not stay for a long time in the set $|\sigma| \leq 2/\sqrt{3}$. The simulation shown in figure 1 illustrates this behavior. Note that as long as $|\sigma| \leq 2/\sqrt{3}$ the estimation error e_2 does not converge, but as soon as $|\sigma| > 2/\sqrt{3}$ the convergence is fast.

6 Conclusions

A new method for the design of nonlinear observers for a class of systems with bad trajectories has been proposed. The class of system's trajectories for which convergence is assured is characterized in terms of conditions similar to the persistency of excitation ones of the parameter identification literature. Since this paper represents a proposal, many issues have to be addressed. In particular, the use of LMIs for the design is under study.

References

[1] M. Arcak and P. Kokotovic. Nonlinear observers: A circle criterion design. In *Proceedings of the 38th. Conference on Decision & Control*, pages 4872–4876, Phoenix, Arizona, USA, 1999. IEEE.

[2] G. Besançon and H. Hammouri. On uniform observation of nonuniformly observable systems. *Syst. Control Lett.*, 29:9–19, 1996.

[3] Ch. Cao, A.M. Annaswamy, and A. Kojic. Parameter convergence in nonlinearly parametrized systems. *IEEE Transactions on Automatic Control*, 48(3):397–412, March 2003.

[4] F. Celle, J.-P. Gauthier, D. Kazakos, and G. Sallet. Synthesis of nonlinear observers: A harmonic-analysis approach. *Math. Syst. Theory*, 22:291–322, 1989.

[5] X. Fan and M. Arcak. Observer design for systems with multivariable monotone nonlinearities. *Systems & Control Letters*, 50:319–330, 2003.

[6] J.P. Gauthier and I. Kupka. *Deterministic Observation Theory and Applications*. Cambridge University Press, Cambridge, UK, 2001.

[7] H. Hammouri and J. de Leon-Morales. Observer synthesis for state-affine systems. In *Proc. 29th Conference on Decision and Control*, Honolulu, Hawaii, December 1990.

[8] H.K. Khalil. *Nonlinear Systems*. Prentice–Hall, Upsaddle River, New Jersey, third edition, 2002.

[9] R. Marino and P. Tomei. *Nonlinear Control Design; Geometric, Adaptive & Robust*. Prentice Hall, London, 1995.

[10] E.A. Misawa and J.K. Hedrick. Nonlinear observers —a state-of-the-art survey. *Trans. ASME J. Dynamic Syst. Meas. Control*, 111:344–351, 1989.

[11] K.S. Narendra and A.M. Annaswamy. *Stable Adaptive Systems*. Prentice Hall, Englewood Cliffs, N.J., 1989.

[12] H. Nijmeijer and T.I. Fossen, editors. *New Directions in Nonlinear Observer Design*. Number 244 in Lecture notes in control and information sciences. Springer–Verlag, London, 1999.

[13] H. Sussmann. Single input observability of continuous time systems. *Math. Syst. Theory*, 12:371–393, 1979.

[14] J. Tsinias. Observer design for nonlinear systems. *Syst. Control Lett.*, 13:135–142, 1989.

 ISBN: 970-32-2137-8